A POSTERIORI ERROR ESTIMATES
FOR UPWIND FINITE VOLUME SCHEMES
FOR NONLINEAR CONSERVATION LAWS
IN MULTI DIMENSIONS

DIETMAR KRÖNER AND MARIO OHLBERGER

ABSTRACT. In this paper we shall derive *a posteriori* error estimates in the
L^1-norm for upwind finite volume schemes for the discretization of nonlinear
conservation laws on unstructured grids in multi dimensions. This result is
mainly based on some fundamental a priori error estimates published in a
recent paper by C. Chainais-Hillairet. The theoretical results are confirmed
by numerical experiments.

1. INTRODUCTION

One of the most important tools for accelerating comprehensive computations in
multi dimensions (in particular in 3-D) is the local adaption of the grid. In order
to minimize the computing time the local grid size should be chosen such that the
error $||u - u_h||$ between the exact solution u and the numerical solution u_h is less
than a given tolerance and such that the total number of cells is as small as possible.
This can be obtained if an *a posteriori* error estimate of the form

$$(1) \qquad ||u - u_h|| \le c \sum_j \eta_j(h_j) + R_h$$

is given, where $\eta_j(h_j)$ are local quantities related to the diameter h_j of each cell or
edge such that $\eta_j(h_j)$ can be computed if u_h is known, and where R_h is related to
the approximation of the data, with $h := \max_j h_j$. Also the constant c should be
known. If $\eta_j(h_j)$ and R_h become sufficiently small when h_j and h tend to zero, the
right hand side in (1) can be made small by local mesh refinement.

For elliptic and parabolic problems the theory of *a posteriori* error estimates of
the form (1) is well developed [6], [7], [8], [9], [19]. But up to now there are no
analogous results for initial value problems for nonlinear conservation laws of the
form

$$(2) \qquad \partial_t u + \operatorname{div} F(x, t, u) = 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^+,$$

$$(3) \qquad u(x, 0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N.$$

Instead of rigorous error estimators, in many applications error indicators, shock indicators or grid indicators have been used in order to find those regions with steep gradients. Usually these indicators are based on discrete gradients or higher order discrete derivatives of the discrete solution. They are used to control the local process of refining and coarsening the grid. But these indicators give no information about the true error $||u - u_h||$.

In this paper we shall exhibit a rigorous *a posteriori* error estimate of the form (1) in the L^1-norm in space and time. Earlier results related to this topic were published in [2], [3], [4], [5], [10], [14], [15], [17], [18], [20] for the nonlinear case and in [9], [13], [14], [16], [19] for linear systems of equations. In Section 2 we will fix the notation and present the main result. Furthermore we will quote the fundamental estimates from [1] and put them together in order to derive the main result. Numerical examples will be given in Section 3.

2. NOTATION AND MAIN RESULT

In this section we will fix the notations and assumptions, define the upwind finite volume scheme for solving (2) and (3), and present the *a posteriori* estimate, the main result of this paper. For the data we have to assume the following conditions:

(4) $u_0 \in L^\infty(\mathbb{R}^N) \cap BV_{loc}(\mathbb{R}^N)$ with $A, B \in \mathbb{R}$ such that $A \leq u_0 \leq B$ a.e.,

(5) $F \in C^1(\mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^N),$

(6) $\sum_j \dfrac{\partial F_j}{\partial x_j}(x, t, s) = 0$ for all $(x, t, s) \in \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}.$

For all compact sets $K \subset \mathbb{R}$ there exists a constant $c_0(K)$ such that

(7) $|\partial_s F(x, t, s)| \leq c_0(K),$
(8) $|\partial_x \partial_s^\beta F(x, t, s)| + |\partial_t \partial_s^\beta F(x, t, s)| \leq c_0(K)$

for almost all $(x, t, s) \in \mathbb{R}^N \times \mathbb{R}^+ \times K$ and $\beta \in \{0, 1\}$.

Remark 2.1. Under the above assumptions above the existence of an entropy solution of (2) and (3) is proved in Theorem 5 of [1]. The uniqueness follows from Theorem 6 of [1].

For $t_n \in \mathbb{R}^+$ let $\mathcal{T}^n = \{T_j | j \in I^n\}$ be a mesh of $\mathbb{R}^N$ such that the interface of two neighbouring cells T_j, T_l of $\mathcal{T}^n$ is included in a hyperplane (see also [1]). The joint edge of T_j and T_l will be denoted by S_{jl}. Let $h_j := \operatorname{diam} T_j$ and $h_{\min}^n := \min_{j \in I^n} h_j$. We assume that $h_{\min}^n > 0$ and that there exists an $\alpha > 0$ such that for all $h_j > 0$ we have

(9)
$$\alpha h_j^N \leq \operatorname{meas}(T_j),$$
$$\alpha \operatorname{meas}(S_{jl}) \leq h_j^{N-1}$$

for all n and for all $j, l \in I^n$. Define $h_{jl} := \max\{\operatorname{diam} T_j, \operatorname{diam} T_l\}$, which gives us the estimate $|S_{jl}| \leq h_{jl}^{N-1}$. Of course T_j, S_{jl}, h_j and h_{jl} depend on n, but for simplicity we suppress the index n here.

Remark 2.2. In the situation where $\triangle t^n = t^{n+1} - t^n$ and the $\mathcal{T}^n$ are constant in time and h_j can be replaced by $h := \max_j \operatorname{diam}(T_j)$, the condition (9) corresponds

exactly to condition (5) in [1]. Thus all results of [1] hold in this more restrictive situation.

Nevertheless we have verified that all results of [1] used in this article to prove the *a posteriori* error estimates hold true in the more general situation (9).

For any $j, l \in I^n$ and $t_n \in \mathbb{R}^+$ there is a numerical flux $g_{jl}^n : \mathbb{R}^2 \to \mathbb{R}$ which satisfies the following conditions for all $u, v, u', v' \in [A, B]$:

(10)   The numerical flux $g_{jl}^n(u, v)$ is monotone increasing with respect to u

and monotone decreasing with respect to v.

Furthermore there exists a constant L, independent of j, l, n and h, such that for all u, v, u', v' as above

$$(11) \qquad g_{jl}^n(u, v) = -g_{lj}^n(v, u),$$

$$(12) \qquad |g_{jl}^n(u, v) - g_{jl}^n(u', v')| \leq L|S_{jl}|(|u - u'| + |v - v'|)$$

and

$$(13) \qquad g_{jl}^n(u, u) = \frac{1}{\Delta t^n} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} F(x, t, u)\, n_{jl}\, dx dt,$$

where Δt^n is the time step, $t^n := \sum_{l=1}^n \Delta t^l$ and n_{jl} is the outer unit normal to S_{jl} with respect to T_j.

From the assumption (8), (13) we obtain the following fact.

Lemma 2.3. *There exists a constant M_1 such that*

$$(14) \qquad |F(x, t, v)n_{jl} - \frac{1}{|S_{jl}|} g_{jl}^n(v, v)| \leq M_1(h_{jl} + \Delta t^n).$$

Proof. It follows from the assumptions (8), (13). $\qquad\qquad\square$

Remark 2.4. If $F(x, t, v) = F(v)$, then the condition (13) implies that $M_1 = 0$ and that the left-hand side in (14) is equal to zero.

Now the upwind finite volume scheme for computing approximate solutions to (2), (3) is defined by

Definition 2.5 (Finite volume scheme). Let

$$
(15) \qquad
\begin{aligned}
u_j^0 &:= \frac{1}{|T_j|} \int_{T_j} u_0, \\
u_j^{n+1} &:= u_j^n - \frac{\Delta t^n}{|T_j|} \sum_{l \in N(j)} g_{jl}^n(u_j^n, u_l^n)
\end{aligned}
$$

for all $n \in \mathbb{N}$ and $j, l \in I^n$. Here $N(j)$ denotes the indices of the neighbouring cells of T_j.

For the time step Δt^n we assume the following CFL-condition for given $\xi \in]0, 1[$:

$$(16) \qquad \Delta t^n \leq \frac{(1 - \xi)\alpha^2 h_{\min}^n}{2L},$$

where L is the Lipschitz constant from (12). Let us denote

$$(17) \qquad u_h(x, t) := u_j^n \quad \text{if} \quad x \in T_j (= T_j^n), \quad t^n < t \leq t^{n+1}.$$

Lemma 2.6 ($\mathbf{L}^\infty$ stability). *Assume that the assumptions* (4),...,(13), (16) *are fulfilled. Then the approximate solution u_h defined in* (17) *satisfies*

$$A \le u_j^n \le B, \ \forall \ n \in \mathbb{N}, \ \forall \ j \in I^n,$$

and

$$||u_h(t^n)||_{L^\infty(\mathbb{R}^N)} \le ||u_0||_{L^\infty(\mathbb{R}^N)}, \ \forall \ n \in \mathbb{N}.$$

Proof. Repeat the proof of Lemma 1 in [1], where the global CFL-condition is replaced by the local conditions (9) and (16). $\qquad\square$

Let u be the exact solution of (2), (3) and u_h be the discrete solution as defined in (17). In [1], [2], [20], it was shown that in the uniform case (i.e. h_j in (9) can be replaced by $h := \max_j \operatorname{diam}(T_j)$) under the assumptions (4),...,(13), (16) we have for any compact set $K \subset \mathbb{R}^N \times \mathbb{R}^+$

$$(18) \qquad \int_K |u(x,t) - u_h(x,t)|dxdt \le ch^{\frac{1}{4}},$$

where the constant c depends only on K and the data F, u_0, L, c_0, α and ξ.

Now it turns out that the same tools which have been used for proving (18) in [1] can be used to show an *a posteriori* estimate. In order to present the details we have to define the following measures as in [1].

Definition 2.7. (Measures) For $\phi \in C_0^0(\mathbb{R}^N)$, $\phi \in C_0^0(\mathbb{R}^N \times \mathbb{R}^+)$ respectively let

$$\langle \mu_h^0, \phi \rangle := \int_{\mathbb{R}^N} |u_0(x) - u_h(x,0)| \, \phi(x)dx,$$

$$\langle \lambda_h, \phi \rangle := \sum_n \sum_j |u_j^{n+1} - u_j^n| \int_{t^n}^{t^{n+1}} \int_{T_j} \phi(x,t)dxdt,$$

$$\langle \mu_{jl}^n, \phi \rangle := \frac{1}{(\Delta t^n)^2 |T_j| \, |S_{jl}|} \int_{t^n}^{t^{n+1}} \int_{T_j} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} \int_0^1 (h_{jl} + \Delta t^n)$$
$$(19) \qquad\qquad\qquad \times \phi(\gamma + \theta(x - \gamma), s + \theta(t - s)) \, d\theta dx dt d\gamma ds,$$

$$\langle \nu_{jl}^n, \phi \rangle := \frac{1}{\Delta t^n |S_{jl}|} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} \int_0^1 (h_{jl} + \Delta t^n)^2 \phi(\xi + \theta(\gamma - \xi),$$
$$\tau + \theta(s - \tau)) \, d\theta d\xi d\tau d\gamma ds,$$

$$\langle \mu_h, \phi \rangle := \langle \lambda_h, \phi \rangle + 2 \sum_n \sum_{edges} \Delta t^n (S_{jl}^1 + S_{jl}^2)$$
$$+ 2M_1 \sum_n \sum_{edges} \langle \nu_{jl}^n, \phi \rangle,$$

where $n \in \mathbb{N}$, $\sum_{edges}$ refers to the sum over all edges of the mesh with neighbouring triangles T_j, T_l and

$$(20) \qquad \begin{aligned} S_{jl}^1 &= \max_{u_l^n \le c \le d \le u_j^n} (g_{jl}^n(d,c) - g_{jl}^n(d,d))\langle \mu_{jl}^n, \phi \rangle, \\ S_{jl}^2 &= \max_{u_l^n \le c \le d \le u_j^n} (g_{jl}^n(d,c) - g_{jl}^n(c,c))\langle \mu_{jl}^n, \phi \rangle. \end{aligned}$$

Lemma 2.8. *Assume* (4),...,(13), (16). *Let u and u_h be the exact solution of* (2) *and the discrete solution of* (15) *respectively. Then for all $R > 0$, $T > 0$ there exist*

constants $M_2, M_3 > 0$ such that in the case of a uniform mesh (i.e. h_j in (9) can be replaced by $h := \max_j diam(T_j)$)

$$\mu_h(B_R(0) \times [0,T]) \leq M_2(h + \sqrt{h}),$$
(21)
$$\mu_h^0(B_R(0)) \leq M_3 h$$

uniformly in h.

Proof. See [1], Theorem 4.   □

Also, from Theorem 4 in [1] we obtain

Theorem 2.9. *Assume (4),...,(13), (16). Let u and u_h be the exact solution of (2) and the discrete solution of (15) respectively. Let μ_h, μ_h^0 be defined as in Definition 2.7. Then for all $k \in \mathbb{R}$ and $\phi \in C_0^\infty(\mathbb{R}^N \times \mathbb{R}^+, \mathbb{R}^+)$ we have*

$$\int_{\mathbb{R}^N \times \mathbb{R}^+} |u_h - k|\partial_t \phi + (F(x,t,\max\{u_h,k\}) - F(x,t,\min\{u_h,k\}))\nabla\phi \, dx dt$$
(22)
$$+ \int_{\mathbb{R}^N} |u_0 - k|\phi(x,0)dx$$

$$\geq -\int_{\mathbb{R}^N \times \mathbb{R}^+} (|\partial_t \phi| + |\nabla\phi|)d\mu_h - \int_{\mathbb{R}^N} \phi(x,0)d\mu_h^0.$$

Proof. This follows by integration by parts and using the definition of u_h in (17) and Definition 2.5 (cf. [1], Theorem 4). Note that this part of the proof of Theorem 4 in [1] also holds in the general case of nonuniform grids as described in (9).   □

For the following lemma we need

(23) $\mathcal{M}(\Omega) :=$ the set of positive continuous linear forms on $C^0(\Omega)$.

Lemma 2.10. *Assume (4),...,(7), $u_0 \in BV(\mathbb{R}^N)$, $\tilde{u} \in L^\infty(\mathbb{R}^N \times \mathbb{R}^+), A \leq \tilde{u} \leq B$ a.e., that there exist measures $\mu \in \mathcal{M}(\mathbb{R}^N \times \mathbb{R}^+)$, $\mu^0 \in \mathcal{M}(\mathbb{R}^N)$ such that the statement of Theorem 2.9 holds with $\tilde{u}$ instead of u_h, and that u is the unique solution of (2), (3).*
Let $\omega \in \mathbb{R}^+$ be given, and let $\rho \in C_0^1(\mathbb{R}^+; [0,1])$ be such that $\rho' \leq 0$ and

$$\rho = 1 \quad on \quad [0,R],$$
(24)
$$\rho = 0 \quad on \quad [R+1,\infty]$$

and

$$\psi(x,t) := \frac{T-t}{T}\rho(|x-x_0| + \omega t) \quad on \quad \mathbb{R}^N \times [0,T],$$
(25)
$$\psi(x,t) := 0 \quad on \quad \mathbb{R}^N \times [T,\infty].$$

Then we have

$$\int_{\mathbb{R}^N \times \mathbb{R}^+} |\tilde{u} - u|\partial_t \psi + (F(x,t,\max\{\tilde{u},u\}) - F(x,t,\min\{\tilde{u},u\}))\nabla\psi \, dx dt$$
(26)
$$\geq -a_0\mu^0(\{\psi(.,0) \neq 0\}) - 2\sqrt{bc}\mu(\{\psi \neq 0\})^{\frac{1}{2}} - a\mu(\{\psi \neq 0\}),$$

with

$$a := 2\omega + \frac{1}{T} + 2,$$

$$a_0 := 1,$$

$$b := 4 + 2^{N+2},$$

(27)
$$c := \|u\|_{BV}(2(2\omega + \frac{1}{T}) + (8 + 2^{N+5})c_0([A,B]))$$

$$+ \|u_0\|_{BV}(2^{N+4}c_0([A,B]) + 1)$$

$$+ 2c_0([A,B]) \max\{|A|,|B|\}(|B_{R+1}(0)| - |B_R(0)|)T,$$

where c_0 is the constant in (7) and (8).

Proof. See [1], Lemma 8. To get the constants a, a_0, b and c one has to repeat the arguments in [1] and control all constants which appear in the estimates.   □

Now we can present the *a posteriori* error estimate, the main result of this paper. Let $R, \omega = c_0([A,B])$ (see (7)), and T be given, and let

$$I_0 := \{n | 0 \le t^n \le \min\{\frac{R+1}{\omega}, T\},$$

(28) $$D_{R+1} := \{(x,t) | |x - x_0| + \omega t < R + 1\},$$

$$M(t) := \{j | \text{ there exists } x \in T_j \text{ such that } (x,t) \in D_{R+1}\}.$$

Theorem 2.11. *Assume the conditions are as in Theorem 2.9 and let $u_0 \in BV(\mathbb{R}^N)$. Let $K \subset\subset \mathbb{R}^N \times \mathbb{R}^+$, $\omega = c_0([A,B])$ (see (7)) and choose T, R and x_0 such that $T \in]0, \frac{R}{\omega}[$ and*

(29)
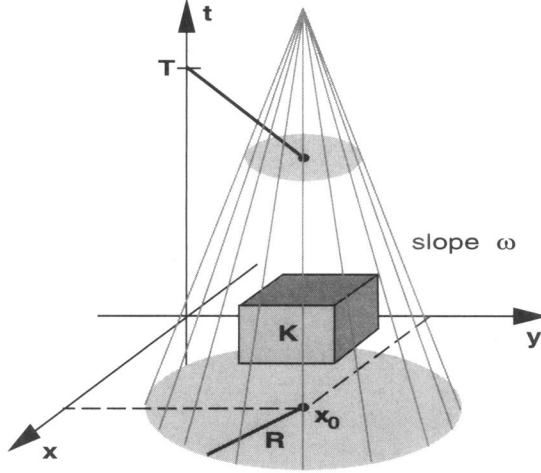$$K \subset \bigcup_{0 \le t \le T} B_{R-\omega t}(x_0) \times \{t\}.$$

Then we have

(30)
$$\int_K |u - u_h| \le T[a_0 \int_{|x-x_0|<R+1} |u_0(x) - u_h(x,0)|dx + aQ + 2\sqrt{bcQ}],$$

where

$$Q := \sum_{n \in I_0} \sum_{j \in M(t^n)} |u_j^{n+1} - u_j^n| \Delta t^n h_j^N$$

$$+ 2 \sum_n \sum_{edges} \Delta t^n (\Delta t^n + h_{jl}) \max_{u_l^n \le c \le d \le u_j^n} (g_{jl}^n(d,c) - g_{jl}^n(d,d))\delta_{jl}^n$$

(31)
$$+ 2 \sum_n \sum_{edges} \Delta t^n (\Delta t^n + h_{jl}) \max_{u_l^n \le c \le d \le u_j^n} (g_{jl}^n(d,c) - g_{jl}^n(c,c))\delta_{jl}^n$$

$$+ 2M_1 \sum_n \sum_{edges} \Delta t^n h_{jl}^{N-1} (\Delta t^n + h_{jl})^2 \delta_{jl}^n$$

and

(32)
$$\delta_{jl}^n := 0 \quad if \quad (T_j \cup T_l) \times [t^n, t^{n+1}] \cap D_{R+1} = \emptyset,$$

$$\delta_{jl}^n := 1 \quad otherwise.$$

FIGURE 1. The relation between ω, R, T and K.

Proof. By assumption we have

$$(33) \qquad K \subset \bigcup_{0 \le t \le T} B_{R-\omega t}(x_0) \times \{t\}.$$

The relation between ω, R, T and K is illustrated in Figure 1.

Because of Lemma 2.6 and Theorem 2.9, Lemma 2.10 holds for $\tilde{u} := u_h$. This means that

$$
\begin{aligned}
(34) \qquad & \int_{\mathbb{R}^N \times \mathbb{R}^+} [|u_h - u|(\frac{T-t}{T}\rho'\omega - \frac{1}{T}\rho) \\
& + (F(., \max\{u_h, u\}) - F(., \min\{u_h, u\}))\frac{T-t}{T}\rho'\frac{x-x_0}{|x-x_0|}]dxdt \\
& \ge -a_0\mu_h^0(\{\psi(.,0) \ne 0\}) - 2\sqrt{bc}\mu_h(\{\psi \ne 0\})^{\frac{1}{2}} - a\mu_h(\{\psi \ne 0\}).
\end{aligned}
$$

Now the terms containing F can be estimated as follows:

$$
\begin{aligned}
(35) \qquad & F(., \max\{u_h, u\}) - F(., \min\{u_h, u\})\frac{T-t}{T}\rho'\frac{x-x_0}{|x-x_0|} \\
& \le c_0(M_0)|\max\{u_h, u\} - \min\{u_h, u\}|\frac{T-t}{T}|\rho'| \\
& \le c_0(M_0)|u_h - u|\frac{T-t}{T}|\rho'| = \omega|u_h - u|\frac{T-t}{T}|\rho'|.
\end{aligned}
$$

Therefore we obtain from (34)

$$
\begin{aligned}
(36) \qquad & \frac{1}{T}\int_K |u - u_h| = \frac{1}{T}\int_K \rho(|x - x_0| + \omega t)|u - u_h| \le \frac{1}{T}\int_{\mathbb{R}^N \times \mathbb{R}^+} \rho|u - u_h| \\
& \le a_0\mu_h^0(\{\psi(.,0) \ne 0\}) + 2\sqrt{bc}\mu_h(\{\psi \ne 0\})^{\frac{1}{2}} + a\mu_h(\{\psi \ne 0\}),
\end{aligned}
$$

since $(x,t) \in K$ implies that $(x,t) \in B_{R-\omega t}(x_0) \times \{t\}$ and $\rho(|x - x_0| + \omega t) = 1$.

Now the theorem follows from the following two lemmata.

Lemma 2.12. *Let ψ be defined as in (25). Then*

$$(37) \qquad \mu_h^0(\{\psi(.0) \neq 0\}) \leq \int_{|x-x_0|<R+1} |u_0(x) - u_h(x,0)|dx.$$

Proof. We have

$$(38) \qquad \mu_h^0(\{\psi(.,0) \neq 0\}) \leq \mu_h^0(\{|x - x_0| + \omega t < R + 1|_{t=0}\})$$

and

$$(39) \qquad \mu_h^0(\{|x - x_0| < R + 1\}) = \int_{|x-x_0|<R+1} |u_0(x) - u_h(x,0)|dx.$$

Lemma 2.13. *Let ψ be defined as in (25). Then*

$$
\begin{aligned}
\mu_h(\{\psi \neq 0\}) \quad \leq \quad & \sum_{n \in I_0} \sum_{j \in M(t^n)} |u_j^{n+1} - u_j^n| \Delta t^n h_j^N \\
& + \ 2 \sum_n \sum_{edges} \Delta t^n (\Delta t^n + h_{jl}) \max_{u_l^n \leq c \leq d \leq u_j^n} (g_{jl}^n(d,c) - g_{jl}^n(d,d)) \delta_{jl}^n \\
(40) \qquad & + \ 2 \sum_n \sum_{edges} \Delta t^n (\Delta t^n + h_{jl}) \max_{u_l^n \leq c \leq d \leq u_j^n} (g_{jl}^n(d,c) - g_{jl}^n(c,c)) \delta_{jl}^n \\
& + \ 2M_1 \sum_n \sum_{edges} (\Delta t^n + h_{jl})^2 \Delta t^n h_{jl}^{N-1} \ \delta_{jl}^n,
\end{aligned}
$$

where δ_{jl}^n is defined in (32).

Proof. The definition of μ_h is given in Definition 2.7. Let us first consider λ_h, which is part of μ_h (see Definition 2.7). Since $D_{R+1} = \{(x,t)| \ |x - x_0| + \omega t < R + 1\}$, we obtain

$$
\begin{aligned}
\lambda_h(\{\psi \neq 0\}) \leq \lambda_h(D_{R+1}) = & \sum_{n \in I_0} \sum_{j \in M(t^n)} |u_j^{n+1} - u_j^n| \int_{t^n}^{t^{n+1}} \int_{T_j} \chi_{D_{R+1}} dx \, dt \\
(41) \qquad \leq & \sum_{n \in I_0} \sum_{j \in M(t^n)} \Delta t^n h_j^N |u_j^{n+1} - u_j^n|.
\end{aligned}
$$

This proves the estimate concerning λ_h. Now we have to estimate μ_{jl}^n, which is also part of μ_h. This measure is defined as (see Definition 2.7)

$$
\begin{aligned}
\langle \mu_{jl}^n, \psi \rangle := & \frac{1}{(\Delta t^n)^2 |T_j| \, |S_{jl}|} \int_{t^n}^{t^{n+1}} \int_{T_j} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} \int_0^1 (h_{jl} + \Delta t^n) \\
& \times \psi(\gamma + \theta(x - \gamma), s + \theta(t - s)) \, d\theta dx dt d\gamma ds.
\end{aligned}
$$

First we consider the case when $T_j \times [t^n, t^{n+1}] \cap D_{R+1} = \emptyset$ and $T_l \times [t^n, t^{n+1}] \cap D_{R+1} = \emptyset$. Since $(\gamma, s) \in T_j \times [t^n, t^{n+1}[$ and $(x, t) \in S_{jl} \times [t^n, t^{n+1}[$, for the convex combination $(\gamma + \theta(x - \gamma), s + \theta(t - s))$ we get

$$(\gamma + \theta(x - \gamma), s + \theta(t - s)) \in T_j \times [t^n, t^{n+1}[,$$

and therefore

$$\psi(\gamma + \theta(x - \gamma), s + \theta(t - s)) = 0.$$

Otherwise

$$|\psi(\gamma + \theta(x - \gamma), s + \theta(t - s))| \leq 1.$$

Hence, using the definition of δ_{jl}^n in (32), we obtain

$$(42) \qquad \langle \mu_{jl}^n, \psi \rangle \leq \delta_{jl}^n (\Delta t^n + h_{jl}).$$

Finally consider the definition of ν_{jl} in Definition 2.7. As before, we get

$$(43) \qquad \langle \nu_{jl}^n, \psi \rangle \leq \delta_{jl}^n (\Delta t^n + h_{jl})^2 \Delta t^n \, h_{jl}^{N-1}.$$

Using the estimates (41), (42) and (43) in the definition of μ_h in (19), we obtain the statement of the lemma. $\qquad\square$

Now the proof of Theorem 2.11 follows if we use (37), (40) in (36). $\qquad\square$

Corollary 2.14. *Under the assumption of Theorem 2.11, and if $F(x, t, v) = F(v)$ (see Remark 2.4), we have*

$$(44) \quad \int_K |u - u_h| \;\; \leq \;\; T\left(a_0 \int_{|x-x_0|<R+1} |u_0(x) - u_h(x, 0)| dx + aQ + 2\sqrt{bcQ}\right),$$

where

$$(45) \qquad Q \;\; := \;\; \sum_{n \in I_0} \sum_{j \in M(t^n)} \Delta t^n h_j^N |u_j^{n+1} - u_j^n|$$

$$+ 4L \sum_n \sum_{E(t_n)} \Delta t^n (\Delta t^n + h_{jl}) h_{jl}^{N-1} |u_j^n - u_l^n|)$$

and $E(t_n)$ is the set of all edges which lie in $M(t^n)$.

Remark 2.15. In the situation of this corollary we have $||u||_{BV} \leq ||u_0||_{BV}$, so that the constant c, as defined in (27), can be calculated a priori. In the general case $F = F(x, t, v)$ there exist, at least in two space dimensions, constants C, C' depending only on F, u_0, Ω, T and $h/\Delta t^n$, such that for all compact sets $\Omega \subset \mathbb{R}^2$

$$||u||_{BV(\Omega \times [0,T[)} \leq T e^{CT} C'.$$

See Lemma 7 of [1] for this result.

3. NUMERICAL TESTS

In this section we want to confirm the theoretical results of Section 2 by some numerical tests in two space dimensions. Therefore we first introduce an adaptive algorithm, which is based on the *a posteriori* error estimate of Corollary 2.14. In this adaptive algorithm we can prescribe an error tolerance for the L^1- error between the exact and the numerical solution of the test problem. Due to the theoretical results of Corollary 2.14 this prescribed error tolerance should be an upper bound for the actual L^1-error, which can be computed in this test examples, where the exact solutions are known. In the following subsections this behaviour is analyzed for two test problems.

3.1. **An adaptive strategy.**

For the definition of an adaptive algorithm let us for simplicity look at the scalar conservation law (2) in the situation, where $F(x,t,v) = F(v)$. Due to the estimate of Corollary 2.14 let us first define the following local estimators:

$$
\begin{aligned}
(\eta_0)_j &= \int_{T_j} |u_0(x) - u_h(x,0)| \, dx, \\
(\eta_t)_j^n &= h_j^2 \, |u_j^{n+1} - u_j^n|, \\
(\eta_x)_{jl}^n &= (\triangle t^n + h_{jl}) \, h_{jl} \, |u_j^n - u_l^n|.
\end{aligned}
$$

Additionally we define the following upper bounds for $\alpha, \beta \in (0,1)$:

$$
\begin{aligned}
B_0 &= \frac{\text{Tol}_0}{a_0 T \bar{M}^0}, \\
B_t &= \min\left\{ \alpha \frac{\text{Tol}_t}{a T^2 \bar{M}^n}, (1-\alpha)^2 \frac{\text{Tol}_t^2}{4bcT^3 \bar{M}^n} \right\}, \\
B_x &= \min\left\{ \beta \frac{\text{Tol}_x}{4LaT^2 \bar{E}^n}, (1-\beta)^2 \frac{\text{Tol}_x^2}{16LbcT^3 \bar{E}^n} \right\}.
\end{aligned}
$$

Here a_0, a, b, c, T and L are the constants of Corollary 2.14. $\bar{M}^n, \bar{E}^n$ denote the number of triangles and edges in $M(t^n), E(t^n)$ respectively, and $\text{Tol}_0, \text{Tol}_t, \text{Tol}_x$ are the tolerances for the estimators η_0, η_t and η_x. Moreover let $\text{Tol}_0, \text{Tol}_t, \text{Tol}_x$ be chosen in such a way that for a prescribed tolerance Tol we have

$$
\text{Tol} = \text{Tol}_0 + \text{Tol}_t + \text{Tol}_x.
$$

With this definitions we can now state the following corollary.

Corollary 3.1. *Let the assumptions of Corollary 2.14 be fulfilled. If in addition $(\eta_0)_j \leq B_0, (\eta_t)_j^n \leq B_t$ and $(\eta_x)_{jl}^n \leq B_x$ for all $n \in I_0$, $j \in M(t^n)$ and all j, l, such that S_{jl} is an edge of $E(t^n)$, then the following estimate holds for any prescribed tolerance Tol :*

$$
\int_K |u - u_h| \leq \text{Tol}.
$$

Proof. Using the assumptions of the corollary, we get, by Corollary 2.14,

$$
\int_K |u - u_h| \leq T\Bigg(a_0 \int_{|x-x_0|<R+1} |u_0(x) - u_h(x,0)|dx
$$

$$
+ a\Bigg(\sum_{n \in I_0} \sum_{j \in M(t^n)} \Delta t^n h_j^2 |u_j^{n+1} - u_j^n|
$$

$$
+ 4L \sum_{n \in I_0} \sum_{S_{jl} \in E(t^n)} \Delta t^n (\Delta t^n + h_{jl}) h_{jl} |u_j^n - u_l^n| \Bigg) \Bigg)
$$

$$
+ 2\sqrt{bc}\Bigg(\sum_{n \in I_0} \sum_{j \in M(t^n)} \Delta t^n h_j^2 |u_j^{n+1} - u_j^n|
$$

$$
+ 4L \sum_{n \in I_0} \sum_{S_{jl} \in E(t^n)} \Delta t^n (\Delta t^n + h_{jl}) h_{jl} \Bigg)^{1/2} |u_j^n - u_l^n| \Bigg)
$$

$$
\leq T\Bigg(a_0 \sum_{j \in M(0)} B_0 + a \sum_{n \in I_0} \sum_{j \in M(t^n)} \Delta t^n B_t + 4La \sum_{n \in I_0} \sum_{S_{jl} \in E(t^n)} \Delta t^n B_x
$$

$$
+ 2\sqrt{bc} \sqrt{\sum_{n \in I_0} \sum_{j \in M(t^n)} \Delta t^n B_t + 4L \sum_{n \in I_0} \sum_{S_{jl} \in E(t^n)} \Delta t^n B_x} \Bigg)
$$

$$
\leq \text{Tol}_0 + \alpha \text{Tol}_t + \beta \text{Tol}_x + \sqrt{(1-\alpha)^2 \text{Tol}_t^2 + (1-\beta)^2 \text{Tol}_x^2}
$$

$$
\leq \text{Tol}.
$$

$\square$

Due to this result the adaptive algorithm reads as follows:

for all $n \in I_0$ {
  for all $j \in M(t^n)$ and all $S_{jl} \in E(t^n)$ {
    coarse T_j and/or T_l if $(\eta_0)_j \leq \theta B_0$, $(\eta_t)_j^n \leq \theta B_t$ or $(\eta_x)_{jl}^n \leq \theta B_x$;
    refine T_j and/or T_l, until $(\eta_0)_j \leq B_0$, $(\eta_t)_j^n \leq B_t$ and $(\eta_x)_{jl}^n \leq B_x$;
    calculate the next time step on the resulting mesh;
  }
}

Here θ is a threshold value, which should be chosen in $(0,1)$ depending on the refinement rules of the mesh.

Remark 3.2. Let us remark that the conditions $(\eta_0)_j \leq B_0$, $(\eta_t)_j^n \leq B_t$ and $(\eta_x)_{jl}^n \leq B_x$ can always be fulfilled, as the estimators converge faster towards zero than the upper bounds do, when h_j, h_{jl} converge to zero.

Furthermore, it is important to state that the time interval Δt^n is also adapted in this algorithm, because Δt^n is directly connected to $h_{\min}$ through the CFL condition.

In the following two subsections we now want to verify the statement of Corollary 3.1 for this adaptive algorithm in two test situations, where the exact solution of the problem is known. Therefore we prescribe an error tolerance Tol and compare the actual error $\int_K |u - u_h|$ to this tolerance.

3.2. A Burgers type problem.

Let us consider the nonlinear conservation law (2) together with the initial condition (3) in $\mathbb{R}^2$, where

$$F(x, t, u) = \begin{pmatrix} u^2 \\ u^2 \end{pmatrix} \qquad \text{and} \qquad u_0(x) = \begin{cases} 2, & \text{if} \quad \frac{x_1 + x_2}{2} - 0.5 \leq 0, \\ 1, & \text{if} \quad \frac{x_1 + x_2}{2} - 0.5 > 0. \end{cases}$$
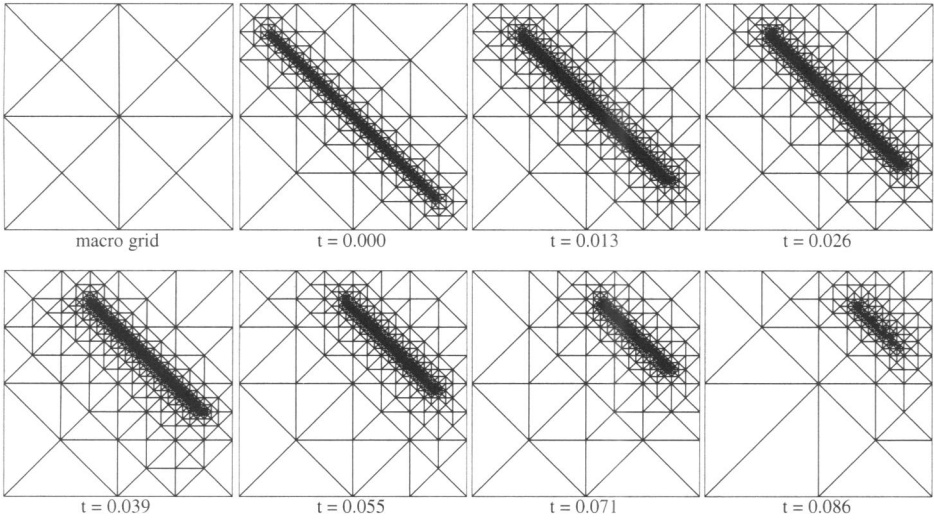
Then the exact solution of this Burgers type problem is

$$u(x, t) = \begin{cases} 2, & \text{if} \quad \frac{x_1 + x_2}{2} - 0.5 \leq 3t, \\ 1, & \text{if} \quad \frac{x_1 + x_2}{2} - 0.5 > 3t. \end{cases}$$

FIGURE 2. Macro grid and adaptive grids at different time steps for the Burgers type problem.
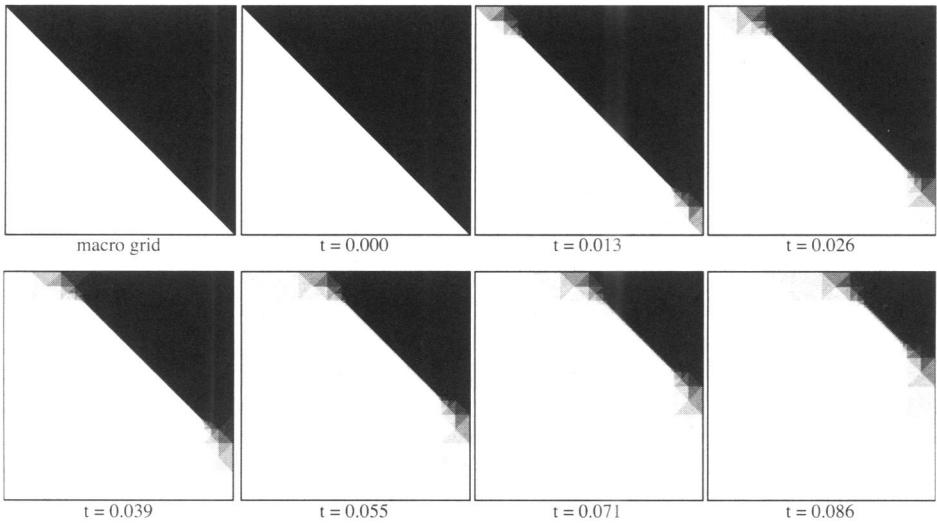

FIGURE 3. The traveling shock of the Burgers type problem.

TABLE 1. Comparison between the prescribed tolerance and the actual error for the Burgers type problem.

TOL	0.024	0.018	0.012	0.008	0.004	0.002	0.001
$\|u - u_h\|_1$	0.01989	0.00975	0.00937	0.00526	0.00147	0.00052	0.00016

In Figure 2 the adaptive refined grids for different time steps are shown. In this example the error should be controlled in the set $B_{0.1}(0.75, 0.75)$. Theorem 2.11 indicates that we have to control the error in the whole cone of dependence. This means we can choose K as the cone of dependence corresponding to $B_{0.1}(0.75, 0.75)$. Due to the choice of the cone K, where the error should be controlled, the grid is only refined inside the cone. The figure shows how the dependence region concentrates to a small disc $B_{0.1}(0.75, 0.75)$ as t reaches $T = 0.1$. Since the last time steps contribute less to the L^1-error over K than the first time steps do, the grid is not that fine at the end of the calculations. The corresponding shock solution is shown in Figure 3. Here the grey values indicate the values of the solution u_h.

Table 1 shows the prescribed tolerances for different simulations of the Burgers type problem and the corresponding actual errors between the exact solution and the discrete solution of the finite volume scheme.

3.3. The rotating cylinder problem.

Let us consider the linear conservation law (2) together with the initial condition (3) in $\mathbb{R}^2$, where

$$F(x, t, u) = \begin{pmatrix} -x_2\, u \\ x_1\, u \end{pmatrix} \quad \text{and} \quad u_0(x) = \begin{cases} 1, & \text{if} \quad |x - (1, 0)^\top| < 0.5, \\ 0, & \text{else}. \end{cases}$$

Then the exact solution of this rotating cylinder problem is

$$u(x, t) = u_0((\cos(t)x_1 + \sin(t)x_2, \ -\sin(t)x_1 + \cos(t)x_2)^\top).$$

The numerical solution of the rotating cylinder problem with the adaptive algorithm is shown in Figure 5, whereas Figure 4 shows the underlying adaptive grid. The set K is chosen such that $K \cap \Omega \times T = B_{0.6}(-1.0, 0.0)$, where $T = 3.15$. Table 2 shows the prescribed tolerances for different simulations of the rotating cylinder problem and the corresponding actual errors between the exact solution and the discrete solution of the finite volume scheme.

TABLE 2. Comparison between the prescribed tolerance and the actual error for the rotating cylinder problem.

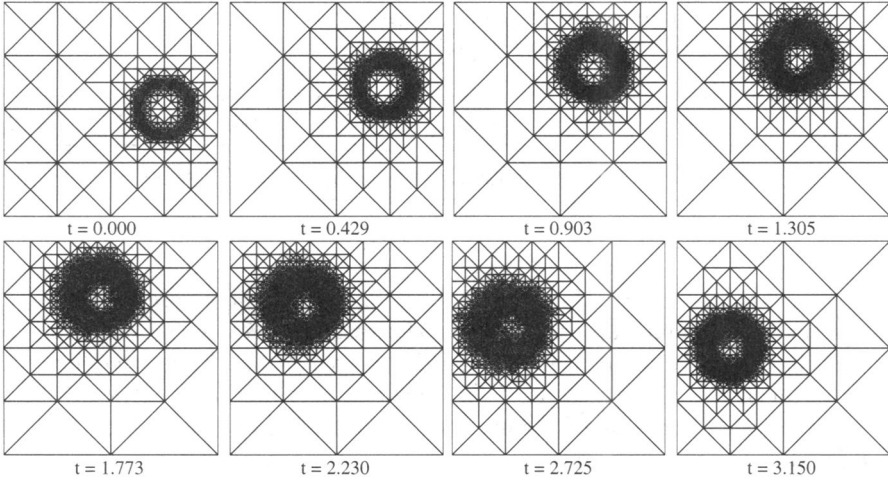| TOL	3.0	2.0	1.6	1.2	0.8	0.4	0.2
$\|u - u_h\|_1$	2.28430	1.54125	1.12592	0.878921	0.61415	0.33538	0.17748

FIGURE 4. Macro grid and adaptive grids at different time steps for the rotating cylinder problem.
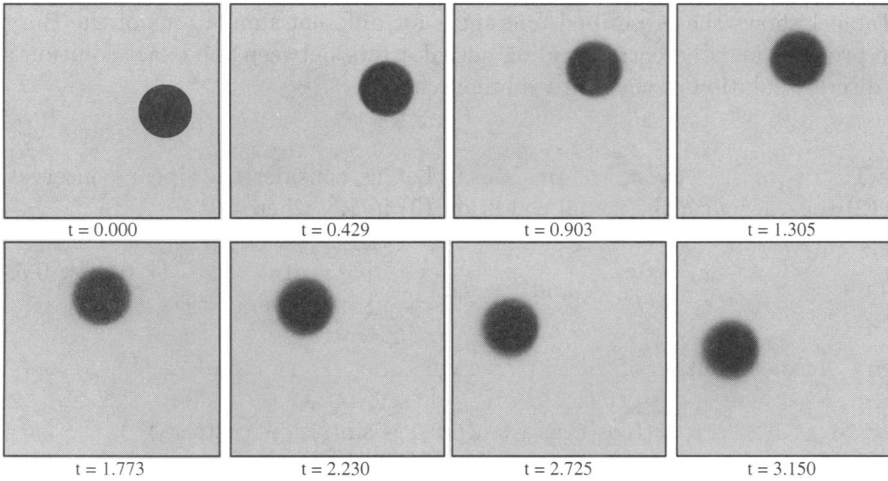

FIGURE 5. The numerical solution of the rotating cylinder problem.

REFERENCES

[1] C. Chainais-Hillairet: Finite volume schemes for a nonlinear hyperbolic equation. Convergence towards the entropy solution and error estimates. Preprint http://umpa.ens-lyon.fr/UMPA/Prepublications/0205/, ENS Lyon (1997).

[2] Cockburn,B. Gremaud, P.A.: An error estimate for finite element methods for scalar conservation laws. IMA Preprint Series No.1144 (1993).

[3] Cockburn,B. Gremaud, P.A.: Error estimates for finite element methods for scalar conservation laws. SIAM J. Numer. Anal., 33 (1996), 522-554. MR **97e**:65096

[4] Cockburn,B., Coquel,F., LeFloch, P.: An error estimate for finite volume methods for multi-dimensional conservation laws. Math. Comput., 63 (1994), 77-103. MR **95d:**65078

[5] Cockburn,B., Coquel,F., LeFloch, P.: An error estimate for high-order accurate finite volume methods for scalar conservation laws. Math. Comput. to appear.

[6] Dörfler, W.: A convergent adaptive algorithm for Poisson's equation. SIAM J. Numer. Anal., 33 (1996), 1106-1124. MR **97e:**65139

[7] Eriksson, K., Johnson,C.: Adaptive finite element methods for parabolic problems I: A linear model problem. SIAM J. Numer. Anal., 28 (1991), 43-77. MR **91m:**65274

[8] Eriksson, K., Johnson,C.: Adaptive finite element methods for parabolic problems IV: Non-linear Problems. SIAM J. Numer. Anal., 32 (1995), 1729-1749. MR **96i:**65081

[9] Houston, P., Süli, E.,: Adaptive Lagrange-Galerkin methods for unsteady convection-dominated diffusion problems. Oxford University Computing Laboratory, Numerical Analysis Group, Report Nr. 95/24.

[10] Johnson,C., Szepessy, A.: Adaptive finite element methods for conservation laws based on a posteriori error estimates. Commun. Pure Appl. Math., 48 (1995), 199-234. MR **97b:**76084

[11] Kröner, D., Noelle, S., Rokyta, M.: Convergence of higher order upwind finite volume schemes on unstructured grids for scalar conservation laws in several space dimensions. Numer. Math., 71 (1995), 527-560. MR **97j:**65087

[12] Kröner, D., Rokyta, M.: Convergence of upwind finite volume schemes for scalar conservation laws in two dimensions. SIAM J. Numer. Anal., 31 (1994), 324-343. MR **95e:**65085

[13] Kröner, D., Rokyta, M.: A priori error estimates for upwind finite volume schemes in several space dimensions. Preprint 37, Freiburg, Mathematische Fakultät (1996). To appear in SIAM J. Numer. Anal..

[14] Mackenzie, J.A., Süli, E., Warnecke, G.: A posteriori analysis for Petrov-Galerkin approximations of Friedrichs systems. Oxford University Computing Laboratory, Numerical Analysis Group, Report Nr. 95/01.

[15] Noelle, S.: Convergence of higher order finite volume schemes on irregular grids. Adv.Comput.Math., 3 (1995), 197-218. MR **96a:**65137

[16] Süli, E., Houston, P.: Finite element methods for hyperbolic problems: aposteriori analysis and adaptivity. In: Duff, I. S. (ed.) et al., The state of the art in numerical analysis, Clarendon Press, Oxford (1997), 441-471. CMP 98:14

[17] Tadmor, E.: Local error estimates for discontinuous solutions of nonlinear hyperbolic equations. SIAM J. Numer. Anal., 28 (1991), 891-906. MR **92d:**35190

[18] Tveito, A., Winther, R.: On the rate of convergence to equilibrium for a system of conservation laws with a relaxation term. SIAM J. Math. Anal., 28 (1997), 136-161. MR **98a:**35087

[19] Verfürth, R.: Robust a posteriori error estimators for the singularly perturbed Helmholtz equation. Fakultät für Mathematik, Bochum, Bericht Nr. 201 (1996).

[20] Vila, J.P.: Convergence and error estimates in finite volume schemes for general multi-dimensional scalar conservation laws. I Explicit monotone schemes. RAIRO, Modélisation Math. Anal. Numér., 28 (1994), 267-295. MR **96d:**65150

INSTITUT FÜR ANGEWANDTE MATHEMATIK, UNIVERSITÄT FREIBURG, HERRMANN-HERDER-STR. 10, 79104 FREIBURG, GERMANY

E-mail address: `dietmar@mathematik.uni-freiburg.de`

E-mail address: `mario@mathematik.uni-freiburg.de`